Bergen County Mathematics League

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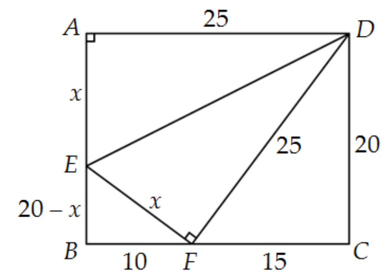
5-1. Since $\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}} = \frac{\frac{x+y}{xy}}{\frac{y-x}{xy}} = \frac{x+y}{y-x} = 2017$, $\frac{x+y}{x-y} = -(\frac{x+y}{y-x}) = -2017$.

5-2. Expanding the left, we have $x^2 + 6xy + 9y^2 + |2x - 7y + 14| = 6xy + 9y^2$, which implies that $x^2 + |2x - 7y + 14| = 0$. Since both terms on the left is nonnegative, we want $x^2 = |2x - 7y + 14| = 0$, from which $x = 0$ and $y = 2$. Therefore, $(x, y) = \boxed{(0, 2)}$.

5-3. Since $i^{n+1} + i^{n+2} + i^{n+3} + i^{n+4} = 0$ for every integer n, $i + i^2 + i^3 + \ldots + i^{2017} = (\sum_{k=0}^{503} i^{4k+1} + i^{4k+2} + i^{4k+3} + i^{4k+4}) + i^{2017} = 0 + i^{2017} = i^{2017} = \boxed{i \text{ or } \sqrt{-1}}$.

5-4. As shown at the right, $\triangle AED \cong \triangle FED$, from which $AD = FD = 25$. Since $CD = 20$, by Pythagorean Theorem $CF = 15$ and $BF = BC - FC = 25 - 15 = 10$. Let $AE = FE = x$, so $EB = AB - AE = 20 - x$. Applying the Pythagorean Theorem on $\triangle BEF$, $(20 - x)^2 + 10^2 = x^2$, from which $x = 12.5$. Therefore, The area of trapezoid $BCDE$ is $\frac{1}{2}(7.5 + 20)(25) = \boxed{343.75}$.



5-5. Let $x - 1 = \frac{y+7}{2} = \frac{z+2}{4} = k$, from which $x = k + 1$, $y = 2k - 7$, and $z = 4k - 2$. Therefore, we have $x^2 + y^2 + z^2 = (k + 1)^2 + (2k - 7)^2 + (4k - 2)^2 = 21k^2 - 42k + 54$. Its minimum value occurs when $k = 1$, so the answer is $\boxed{33}$.

5-6. Let $[x] = n$ and $x = n + f$ for which $0 \le f < 1$. We have $\frac{n}{f} = \frac{n+f}{n}$. Clearly, both n and f must be positive. Clearing fractions and moving everything to one side, we get $n^2 - nf - f^2 = 0$. Solving for n and taking the positive root, $n = f(\frac{1+\sqrt{5}}{2})$. Since $0 < f < 1$, it follows that $n < \frac{1+\sqrt{5}}{2} < 2$. Since n is positive, $n = 1$. We now solve $1 = f(\frac{1+\sqrt{5}}{2})$, from which $f = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$ and $x = n + f = 1 + \frac{\sqrt{5}-1}{2} = \boxed{\frac{1+\sqrt{5}}{2}}$. [Note: The answer is known as the *Golden Ratio*.]