Bergen County Mathematics League

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Brief Contest Solutions #6 **2015-2016**

6-1. Let a_1, a_2, and a_3 be the first three terms of the sequence. We have $(a_1)(a_2)(a_3) = a_2$, which implies that $(a_1)(a_3) = 1$, so $a_1 = a_3 = \boxed{1 \text{ or } -1}$.
[Note: Similarly, we see that $a_1 = a_3 = a_5 = \ldots = a_{2015} = 1$ or -1. Also, $a_2 = a_4 = a_6 = \ldots = a_{2016} = 1$ or -1. Thus the terms of this sequence can be all 1's, or all -1's, or an alternation between 1 and -1, with the first term being either 1 or -1.]

6-2. In any triangle, the sum of the lengths of any two of its sides is greater than the length of the third side, and the difference between the lengths of any two of its sides is less than the length of the third side. Therefore, the least possible length of the triangle's third side is $20 - 16 + 1 = 5$, and the greatest possible length of the third side is $20 + 16 - 1 = 35$. From 5 to 35 inclusive there are a total of $\boxed{31}$ integers.

6-3. Note that $(1 - i)^2 = -2i$ and $256 = (-2i)^8 = ((1 - i)^2)^8 = (1 - i)^{16}$. Therefore, $n = \boxed{16}$.

6-4. If the length of a side of the an equilateral triangle is s, then the area of the triangle is $\frac{s^2\sqrt{3}}{4}$. Decompose the triangle into 3 (non-equilateral triangles) by connecting the vertices to the interior point, so area $= \frac{1}{2}s(1 + 4 + 7) = 6s$. Since $\frac{s^2\sqrt{3}}{4} = 6s$, $s = \boxed{\frac{24}{\sqrt{3}}}$ or $8\sqrt{3}$.

6-5. Since $\log_2 \sin x$ and $\log_2 \cos x$ are both defined, both $\sin x$ and $\cos x$ are positive. Therefore, x is in Quadrant I. We have $0 < x < \pi/2$, from which $0 < 2x < \pi$. The equation is $\log_2 2 \sin x \cos x = -1 \Leftrightarrow \sin(2x) = \frac{1}{2}$. So $2x = \frac{\pi}{6}, \frac{5\pi}{6}$. Solving, $x = \boxed{\frac{\pi}{12}, \frac{5\pi}{12}}$.

6-6. We have $s^2 + 15 = w(w + 7)$, where s and w are positive integers. Both sides of the equation represent the number of participants, and $w^2 + 7w - (s^2 + 15) = 0$. The discriminant, $4s^2 + 109$, must be a perfect square, so $4s^2 + 109 = y^2$ for some integer y. Rearranging terms, $109 = y^2 - 4s^2 = (y + 2s)(y - 2s)$, so $y + 2s$ and $y - 2s$ are factors of 109. Since 109 is prime, its only factors are 109 and 1. Since $s > 0$, we have $y + 2s = 109$ and $y - 2s = 1$. Solving these equations simultaneously, we get $s = 27$. Thus, the number of participants is $s^2 + 15 = \boxed{744}$. [Note: $w = 24$, and $744 = 27^2 + 15 = 24(24 + 7)$.]