Bergen County Mathematics League

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Brief Contest Solutions #3 — 2015-2016

3-1. From the first statement, we know that $x \le 20.15ab\ldots$, where a and b are digits. Clearly, $a \le 4$ and $b \le 9$. Rounding $20.1549\ldots$ to 3 decimal places, we get $\boxed{20.155}$.

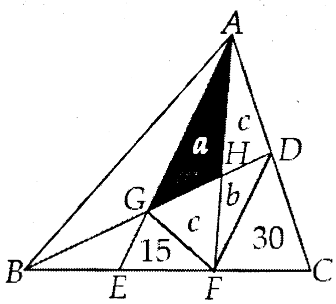
3-2. Calling the numbers a and b: $a + b = 14$ and $ab = 4$. Hence $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{14}{4} = \boxed{\frac{7}{2}}$.

3-3. Since $m\angle B - m\angle A = m\angle C - m\angle B$, we get $2m\angle B = m\angle A + m\angle C$. Thus, $m\angle A + m\angle B + m\angle C = 3m\angle B = 180$, so $m\angle B = \boxed{60 \text{ or } 60°}$.

3-4. Square both sides to get $163 - 56\sqrt{3} = 3a^2 + b^2 - 2ab\sqrt{3}$. Equating coefficients of like terms, $3a^2 + b^2 = 163$ and $2ab = 56$. By simple guess and check, or by algebraic substitution, $(a, b) = \boxed{(7, 4)}$.

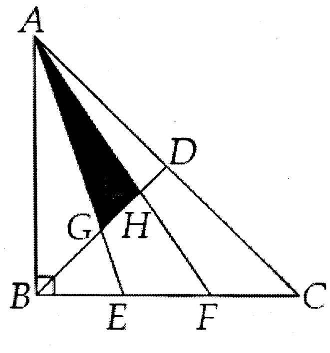
3-5. Dividing both sides of the equation by B, we get $AB \times A = 111$. Since $37 \times 3 = 111$, we have $(A, B) = \boxed{(3, 7)}$.

3-6. Using area, we get $[AEC] = 180 \times \frac{2}{3} = 120$. Draw $\overline{DF}$ and $\overline{FG}$. Note that $\overline{DF}$ is a mid-segment (midline) of $\triangle AEC$. Thus $[DCF] = \frac{1}{4}[AEC] = 30$. Since $ADFG$ is a trapezoid, with $\overline{AG} \parallel \overline{DF}$, we get $[ADF] = [GDF]$, thus $[ADH] = [GFH]$. Let $[AGH] = a$, $[DFH] = b$, and $[ADH] = [GFH] = c$. Since $b + c = [ADF] = [DCF] = 30$, it follows that $[GEF] = \frac{1}{2}([BCD] - [CDGF]) = \frac{1}{2}[\frac{1}{2}(180) - (30 + b + c)] = 15$, and $a + c = [AEF] - [GEF] = 180 \times \frac{1}{3} - 15 = 45$.



Due to same base/height, the proportion $\frac{GH}{HD} = \frac{a}{c} = \frac{c}{b}$ implies that $\frac{a+c}{c+b} = \frac{a}{c} = \frac{45}{30} = \frac{3}{2}$, from which $2a = 3c$. Substitute into $a + c = 45$ to get that $a = \boxed{27}$.

Alternatively, linear transformations scale area so the ratio of areas of corresponding (not similar) regions remains unchanged. We can transform the triangle into a 6×6 right triangle with vertices at $A(0, 6)$, $B(0, 0)$, and $C(6, 0)$, then use proportion to get the area for the larger triangle (with area 180). Using coordinate geometry, we can write the equation of every line segment and determine that the coordinates of G and H are (1.5, 1.5) and (2.4, 2.4) respectively. By using the same base/height, we see that $\frac{[AGH]}{[ABD]} = \frac{GH}{BD} = \frac{0.9\sqrt{2}}{3\sqrt{2}} = \frac{3}{10}$.



Thus, $\frac{[AGH]}{[ABC]} = \frac{3}{20} = \frac{x}{180}$. Solving, we get $x = 27$.